



Differential Geometry II - Smooth Manifolds

Winter Term 2025/2026

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Exercise Sheet 8

Exercise 1:

Let M be a smooth manifold. Show that if S is an embedded submanifold of M , then the subspace topology on S and the smooth structure on S described in *Theorem 5.9* are the only topology and smooth structure with respect to which S is an embedded (or immersed) submanifold.

[Hint: Use [*Exercise Sheet 7*, *Exercise 5(c)*].]

Exercise 2:

Let M be a smooth manifold. Show that if S is an immersed submanifold of M , then for the given topology on S , there exists only one smooth structure making S into an immersed submanifold.

[Hint: Use [*Exercise Sheet 7*, *Exercise 5(b)*].]

Exercise 3:

- (a) Let M be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_p M$ is in $T_p S$ if and only if there exists a smooth curve $\gamma: J \rightarrow M$ whose image is contained in S , and which is also smooth as a map into S , such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.
- (b) Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma: J \rightarrow M$ be a smooth curve whose image happens to lie in S . Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)} S$ of $T_{\gamma(t)} M$.
- (c) Give a counterexample to part (b) if $S \subseteq M$ is immersed but not embedded.

[Hint: Example 4.5(2).]

Exercise 4:

- (a) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Show that if $\Phi: U \rightarrow N$ is a local defining map for S , then it holds that

$$T_p S \cong \ker(d\Phi_p: T_p M \rightarrow T_{\Phi(p)} N) \quad \text{for every } p \in S \cap U.$$

- (b) Let M be a smooth manifold. Suppose that $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi_1, \dots, \Phi_k): M \rightarrow \mathbb{R}^k$. Show that a vector $v \in T_p M$ is tangent to S if and only if $v\Phi_1 = \dots = v\Phi_k = 0$.

Exercise 5 (to be submitted by Thursday, 13.11.2025, 16:00):

- (a) Consider the smooth function

$$\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 - y^2.$$

Show that the level set $\Phi^{-1}(0)$ is an immersed submanifold of \mathbb{R}^2 .

[Hint: View $\Phi^{-1}(0)$ as the disjoint union of two lines (where we remove the origin from one of them).]

- (b) Consider the smooth function

$$\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 - y^3.$$

Show that the level set $\Psi^{-1}(0)$ is not an immersed submanifold of \mathbb{R}^2 .

[Hint: Argue by contradiction and use *Exercise 3(a)*.]